**Math 53: Multivariable Calculus** **Sections 201, 204**

# Worksheet answers for 2021-10-27

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

## Answers to conceptual questions

**Question 1.** The orientation of $C$ is encoded in the unit tangent $\mathbf{T}$; at every point there are two choices of such a unit tangent, corresponding to the two directions in which one can trace out the curve $C$.

Note that $\int_C \mathbf{F} \cdot \mathbf{T}\, \mathrm{d}s$ is typically not useful for actually evaluating work integrals.

## Answers to computations

**Problem 1.**

(a) Letting $P, Q$ denote the components of $\mathbf{F}$ as usual, we verify that $Q_x - P_y = 0$:

$$xh'(x^2+y^2)2y - yh'(x^2+y^2)2x = 0.$$

Provided that the domain of $\mathbf{F}$ is all of $\mathbb{R}^2$ (which it will be, as long as $h(t)$ behaves nicely for all $t \geq 0$), it follows that $\mathbf{F}$ is conservative, because $\mathbb{R}^2$ is simply connected.

(b) We want to find $f$ such that

$$f_x(x,y) = xh(x^2+y^2)$$
$$f_y(x,y) = yh(x^2+y^2).$$

Integrating the first equation with respect to $x$ (using the substitution $u = x^2+y^2$, $\mathrm{d}u = \mathrm{d}x$) gives

$$f(x,y) = \frac{1}{2}g(x^2+y^2) + C(y)$$

where $C(y)$ is any function of $y$. Differentiating this with respect to $y$ we get

$$f_y(x,y) = yh(x^2+y^2) + C'(y)$$

hence we have $C'(y) = 0$ by comparison to our original observations. Thus $C(y)$ is a constant, and a function such as $f(x,y) = \frac{1}{2}g(x^2+y^2) + 42$ will suffice.

(c) The argument in (b) will produce a potential function that is defined on the same domain as $\mathbf{F}$ (namely, the plane minus the origin). However, the argument in (a) is no longer conclusive, because the domain of $\mathbf{F}$ is not simply connected. (It does show that $\mathbf{F}$ is conservative on any simply connected subset of $\mathbb{R}^2 - (0,0)$, though.)

**Problem 2.**

(a) $C$ can be parametrized as $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$. The integral becomes

$$\int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, \mathrm{d}t = \int_0^{2\pi} \mathrm{d}t = \boxed{2\pi}.$$

(b) Following the same procedure as in 1(b), one might find either of the functions

$$f(x,y) = \arctan(y/x) + \text{constant, or } f(x,y) = -\arctan(x/y) + \text{constant}.$$

Notice that the first is not defined when $x = 0$, while the second is not defined when $y = 0$. In either case, the domain of $f$ is *strictly smaller* than that of $\mathbf{F}$.

(c) No, because (a) already showed that $\mathbf{F}$ cannot be conservative on (any region containing) the circle $C$. Remember that to be conservative on some region, the work line integral over *any closed loop* in that region must be equal to zero.

(d) Although the vector field $\mathbf{F}$ is not conservative on its entire domain, it is conservative on the region

$$\{(x,y) \in \mathbb{R}^2 : y \neq 0 \text{ or } x < 0\}.$$

This is the plane minus the nonnegative $x$-axis, which is a simply connected region. It is possible to exhibit a potential function $f$ on this region whose gradient is $\mathbf{F}$:

$$f(x, y) = \text{“}\theta\text{”} = \begin{cases} -\arctan(x/y) + \pi/2 & \text{if } y > 0, \\ \arctan(y/x) + \pi & \text{if } x < 0, \\ -\arctan(x/y) + 3\pi/2 & \text{if } y < 0. \end{cases}$$

This function assigns to each point $(x, y)$ its polar angle $\theta$, measured strictly between 0 and $2\pi$.

Hence by the FTLI, the answer to this question is $f(3, -3) - f(1, 1) = 7\pi/4 - \pi/4 = \boxed{3\pi/2}$.